THE CHANGE OF UNKNOWN CONSTANTS IN SEPARABLE DIFFERENTIAL EQUATIONS

Example 1. *Find the general solution to*

$$\frac{dT}{dt} = -k(T - T_m). \tag{1}$$

Solution. We first separate the variables and get

$$\frac{dT}{T - T_m} = -kdt \Longrightarrow \int \frac{dT}{T - T_m} + C_2 = \int -kdt + C_1.$$

Here C_1, C_2 are two arbitrary constants. This is equivalent to

$$\int \frac{dT}{T - T_m} = \int -kdt + C_1 - C_2. \tag{2}$$

Since C_1, C_2 can take any value as we want, it is easy to see that the value of $C_1 - C_2$ is also arbitrary. So for notational brevity, we can replace $C_1 - C_2$ by one unknown constant, say C_0, that is we set the notation

$$C_0 = C_1 -_2 .$$

So (2) becomes

$$\int \frac{dT}{T - T_m} = \int -kdt + C_0.$$

Solving it, we obtain

$$T(t) = \pm e^{C_0} e^{-kt} + T_m. \tag{3}$$

Let us look at the expression e^{C_0}. The range of e^x is $(0, \infty)$. Since C_0 is arbitrary, e^{C_0} can take any **positive** value, and thus $\pm e^{C_0}$ can be any **NON-ZERO** number. As before, we can simplify our notation by putting

$$C_3 = \pm e^{C_0},$$

and then (3) becomes

$$T(t) = C_3 e^{-kt} + T_m, \quad C_3 \neq 0. \tag{4}$$

Note that by the above argument, $C_3 = \pm e^{C_0}$ cannot be zero, as the exponential function cannot take value 0.

But if we actually set $C_3 = 0$ in (4), we will get

$$T(t) \equiv T_m.$$

Let us verify if this is a solution to (1). Indeed, $\frac{dT}{dt} \equiv 0$ as $T(t) \equiv T_m$ is a constant function. Meanwhile, $-k(T - T_m) = -k(T_m - T_m) = 0$. $T(t) \equiv T_m$ is actually a solution to (1).

Thus it is in fact admissible to take $C_3 = 0$ in (4).

Therefore, the general solution to (1) is

$$T(t) = C_3 e^{-kt} + T_m, \quad C_3 \text{ arbitrary constant.}$$

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Remark 2.

- *The rigorous argument to show that the constant C_3 can actually take value 0 is not required for this course. So you can always omit this step in your computations and assume C_3 is arbitrary.*
- *Students need to be cautious while combining different unknown constants. For example, in the general solution to the falling object problem*

  $$y(t) = \frac{1}{2}gt^2 + C_1 t + C_2,$$

  the constants C_1, C_2 cannot be combined into one, since both

  $$C_1 t$$

  and

  $$C_2$$

  are different from the polynomial

  $$C_1 t + C_2.$$

  However, if we have an expression

  $$C_1 t + C_2 + C_3 t + C_4,$$

  we can set

  $$C_5 = C_1 + C_3, \quad C_6 = C_2 + C_4,$$

  and then reduce the original expression to

  $$C_5 t + C_6.$$

- *While setting $C_3 = \pm e^{C_0}$ to abbreviate the notations, one can be flexible with the choice of notations. For example, to abbreviate the expression $\pm e^C$, we can usually write*

  $$C = \pm e^C.$$

  It might be confusing to use the same notation to replace an old one. But you should get used to this convention, because by following this convenient we can avoid keeping tracking of the subscripts of the unknown constants.